

Critical gravity as van Dam-Veltman-Zakharov discontinuity in anti de Sitter space

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Abstract

We consider critical gravity as van Dam-Vletman-Zakharov (vDVZ) discontinuity in anti de Sitter space. For this purpose, we introduce the higher curvature gravity. This discontinuity can be confirmed by calculating the residues of relevant poles explicitly. For the non-critical gravity of $0 < m_2^2 < -2\Lambda/3$, the scalar residue of a massive pole is given by $2/3$ when taking the $\Lambda \rightarrow 0$ limit first and then the $m_2^2 \rightarrow 0$ limit. This indicates that the vDVZ discontinuity occurs in the higher curvature theory, showing that propagating degrees of freedom is decreased from 5 to 3. However, at the critical point of $m_2^2 = -2\Lambda/3$, the tensor residue of a massive pole blows up and scalar residue is $-5/36$, showing the unpromising feature of the critical gravity.

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1 Introduction

There has been much interest in the massless limit of the massive graviton propagator [1, 2, 3, 4, 5]. A key issue of this approach is that van Dam-Veltman-Zakharov (vDVZ) discontinuity [6] is peculiar to Minkowski space, but it seems unlikely to arise in (anti) de Sitter space. The vDVZ discontinuity implies that the limit of $M_{\text{MP}}^2 \rightarrow 0$ does not yield a massless graviton at the tree level such that the Einstein gravity (general relativity: GR) is isolated from the massive gravity. One has usually introduced the Fierz-Pauli mass term with mass squared M_{FP}^2 [7] for this purpose.

If the cosmological constant (CC, Λ) was introduced and a smooth $M_{\text{FP}}^2/\Lambda \rightarrow 0$ limit exists, first taking the $M_{\text{FP}}^2 \rightarrow 0$ limit (and then, the $\Lambda \rightarrow 0$ limit) recovers a massless graviton, leading to no vDVZ discontinuity in the Einstein gravity. It is worth noting that $M_{\text{FP}}^2 \rightarrow 0$ and $\Lambda \rightarrow 0$ limits do not commute. First taking the $\Lambda \rightarrow 0$ limit, one encounters the vDVZ discontinuity [3]. Another resolution to the discontinuity is possible to occur even in Minkowski space, if the Schwarzschild radius of the scattering objects is taken to be the second mass scale [8]. However, these all belong to the linearized (tree) level calculations. If one-loop graviton vacuum amplitude is computed for a massive graviton [9], the discontinuity appears again. This means that the apparent absence of the vDVZ discontinuity may be considered as an artifact of the tree level approximation. Also, there was the Boulware-Deser instability which states that at the non-linearized level, a ghost appears in the massive gravity theory [10].

On the other hand, critical gravities were recently investigated in the AdS space as candidates of quantum gravity [11, 12, 13, 14]. In the framework of GR with higher curvature terms (namely, higher curvature gravity), the critical gravity is determined by two conditions of

$$\alpha = -3\beta, \quad \beta = -1/2\Lambda. \quad (1)$$

At the critical point, the graviton tensor $h_{\mu\nu}^{\text{cr}}$ satisfies the fourth-order equation under the transverse and traceless gauge and its solution takes the log-form. The non-unitarity issue of log-gravity is not still resolved, indicating that the log-gravity suffers from the ghost problem [15, 16]. To resolve this ghost problem, it was suggested that imposing unitarity may require suppressing the log-modes by choosing an appropriate boundary condition, leaving the Einstein gravity in the IR region [11, 17].

In this work, we wish to regard the appearance of critical gravity as the vDVZ discontinuity of the higher curvature gravity in the AdS_4 space. This picture may be clear when reminding that the critical gravity was originated from the chiral point in cosmological topologically massive gravity in three dimensions: a negative-energy massive graviton

disappears at the chiral point of $\mu = 1/\ell$ and thus, it becomes a massless left-handed mode of the 3D Einstein gravity [18]. It is well known that the 3D Einstein gravity is a gauge theory, which means that there is no propagating degrees of freedom (DOF). This cosmological topological massive gravity at the critical point (CCTMG) may be described by using the logarithmic conformal field theory (LCFT) [19, 20] even for the zero central charge $c_L = 0$. Another massive generalization of the 3D Einstein gravity was proposed by adding a specific quadratic curvature term to the Einstein-Hilbert action [21, 22]. This gravity theory became known as new massive gravity (NMG). This theory was designed for reproducing the ghost-free Fierz-Pauli action for a massive propagating graviton in the linearized level. In this sense, the NMG is the non-linear realization of a massive graviton. Unlike the TMG, the NMG preserves parity. As a result, the gravitons acquire the same mass for both helicity states, indicating 2 DOF. At the critical point of $m^2 = 1/2\ell^2$, two massive modes turned out to be massless left/right-handed modes of the 3D Einstein gravity in the AdS_3 space [23]. Also, this corresponds to zero central charges $c_{L/R} = 0$ on the boundary CFT. We note that *the critical gravity is just the higher dimensional extension of the NMG at the critical point.*

It was argued that there is no vDVZ discontinuity in GR with higher curvature term (for example, $R - 2\Lambda + \beta R^2$) in the AdS_4 space [24]. Recently, a similar analysis including higher curvature terms was performed in D -dimensional anti-de Sitter space, including the NMG [25]. We comment that the vDVZ discontinuity appeared in $f(R)$ gravity because $f(R)$ gravity means GR with an additional scalar, implying 3 DOF initially [26]. In general, the vDVZ discontinuity was closely related to the decreasing DOF of $5 \rightarrow 3$. Hence, the argument in Ref.[26] may be not justified because unless one includes $\alpha R_{\mu\nu} R^{\mu\nu}$, one might not see the vDVZ discontinuity in the massless limit.

Reminding that the vDVZ discontinuity is the massless limit of the massive gravity in the Minkowski space, the critical gravity appears as the massless limit of higher curvature gravity in the AdS_4 space. Therefore, we insist that the critical gravity may be represented as the vDVZ discontinuity of the higher curvature gravity in the AdS_4 space. In this case, we expect to have a decreasing DOF ($5 \rightarrow ?$) at the critical point.

2 Higher curvature gravity

We start with the higher curvature gravity including a bare cosmological constant Λ_0

$$I = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \{ R - 2\Lambda_0 + \beta R^2 + \alpha R_{\mu\nu} R^{\mu\nu} \} \quad (2)$$

with the gravitational constant $\kappa^2 = 8\pi G$. Here we choose β and α to match with the convention in Ref. [11] and their mass dimensions are $[\alpha] = [\beta] = -2$ with $\alpha < 0$ and $\beta > 0$. Also, we follow the signature of $(-+++)$. In the case of $\Lambda_0 = 0$, the theory is renormalizable and it describes 8 DOF (a massless spin-2 graviton with 2 DOF, a massive spin-2 graviton with 5 DOF, and a massive scalar with 1 DOF) [27]. However, the massive graviton suffers from having ghosts. We note again that $f(R)$ gravity with $\alpha = 0$ has 3 DOF (a massless spin-2 graviton and a massive scalar) without ghost [26].

The equation of motion is given by

$$G_{\mu\nu} + E_{\mu\nu} = 0, \quad (3)$$

where the Einstein tensor $G_{\mu\nu}$ and $E_{\mu\nu}$ are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}, \quad (4)$$

$$E_{\mu\nu} = 2\alpha \left(R_{\mu\rho} R_{\nu}{}^{\rho} - \frac{1}{4} R_{\rho\sigma} R^{\rho\sigma} g_{\mu\nu} \right) + 2\beta R \left(R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) \\ + \alpha \left(\nabla^2 R_{\mu\nu} + \nabla_{\rho} \nabla_{\sigma} R^{\rho\sigma} g_{\mu\nu} - 2 \nabla_{\rho} \nabla_{(\mu} R_{\nu)}{}^{\rho} \right) + 2\beta \left(g_{\mu\nu} \nabla^2 - \nabla_{\mu} \nabla_{\nu} \right) R. \quad (5)$$

In this case, the vacuum solution to (3) is the AdS_4 space whose geometry is expressed in terms of the metric ($\bar{g}_{\mu\nu}$) as

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{\Lambda}{3} (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho}), \quad \bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}, \quad \bar{R} = 4\Lambda = -\frac{12}{\ell^2}. \quad (6)$$

Its line element takes the form

$$ds_{\text{AdS}}^2 = \bar{g}_{\mu\nu} dx^{\mu} dx^{\nu} = - \left(1 + \frac{r^2}{\ell^2} \right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{\ell^2} \right)} + r^2 d\Omega_2^2. \quad (7)$$

We note that the effective CC Λ is equal to the bare CC Λ_0 only in four dimensions.

In order to study the propagation of the metric, we introduce the perturbation around the AdS_4 space

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (8)$$

Hereafter we denote the background value with overbar ($\bar{}$). We expect that the theory describes 8 DOF in the AdS_4 space, too. The linearized equation to Eq.(3) with the external source $T_{\mu\nu}$ takes the form

$$\left[1 + \frac{4\alpha\Lambda}{3} + 8\beta\Lambda \right] \delta G_{\mu\nu}(h) + \alpha \left[\bar{\nabla}^2 \delta G_{\mu\nu}(h) - \frac{2\Lambda}{3} \delta R(h) \bar{g}_{\mu\nu} \right] \\ + (\alpha + 2\beta) \left[\bar{g}_{\mu\nu} \bar{\nabla}^2 - \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} + \Lambda \bar{g}_{\mu\nu} \right] \delta R(h) = T_{\mu\nu}, \quad (9)$$

where the linearized Einstein tensor is given by [25]

$$\delta G_{\mu\nu}(h) = \delta R_{\mu\nu}(h) - \frac{\bar{g}_{\mu\nu}}{2} \delta R(h) - \Lambda h_{\mu\nu}. \quad (10)$$

The linearized Ricci tensor and the linearized scalar curvature take the forms, respectively,

$$\begin{aligned} \delta R_{\mu\nu}(h) &= \frac{1}{2} \left[-\bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + 2\bar{\nabla}^\rho \bar{\nabla}_{(\mu} h_{\nu)\rho} \right], \\ &= \frac{1}{2} \left[\Delta_L^{(2)} h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + 2\bar{\nabla}_{(\mu} \bar{\nabla}^\rho h_{\nu)\rho} \right], \end{aligned} \quad (11)$$

$$\delta R(h) = \bar{g}^{\mu\nu} \delta R_{\mu\nu}(h) - h^{\mu\nu} \bar{R}_{\mu\nu} = \bar{\nabla}^\rho \bar{\nabla}^\sigma h_{\rho\sigma} - \bar{\nabla}^2 h - \bar{\Lambda} h, \quad (12)$$

where the Lichnerowicz operator $\Delta_L^{(2)} h_{\mu\nu}$ takes the form

$$\Delta_L^{(2)} h_{\mu\nu} = -\bar{\nabla}^2 h_{\mu\nu} + \frac{8\Lambda}{3} \left(h_{\mu\nu} - \frac{h}{4} \bar{g}_{\mu\nu} \right). \quad (13)$$

The trace of (9) leads to

$$\left[2(\alpha + 3\beta) \bar{\nabla}^2 - 1 \right] \delta R(h) = T. \quad (14)$$

Acting $\bar{\nabla}^\mu$ to (9) leads to zero, indicating that the Bianchi identity is satisfied in the linearized level. As a result, one finds the source-conservation law

$$\bar{\nabla}^\mu T_{\mu\nu} = 0 \quad (15)$$

which states that $T_{\mu\nu}$ is covariantly conserved with respect to the background metric of AdS_4 space.

At this stage, we note that the linearized equation (9) without external sources $T_{\mu\nu}$ is invariant under linearized diffeomorphisms as

$$\delta_\xi h_{\mu\nu} = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu, \quad (16)$$

because of

$$\delta_\xi \delta G_{\mu\nu}(h) = 0, \quad \delta_\xi \delta R(h) = 0. \quad (17)$$

This implies that divergence and double divergence do not provide any constraint on $h_{\mu\nu}$. Hence, the gauge invariant (physical) quantity is left undetermined by the linearized equation (9). We use the diffeomorphisms to impose the transverse gauge condition

$$\bar{\nabla}^\mu h_{\mu\nu} = \bar{\nabla}_\nu h. \quad (18)$$

Additionally, its divergence is given by

$$\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} = \bar{\nabla}^2 h. \quad (19)$$

Using these conditions, we rewrite the linearized Ricci tensor, scalar curvature, and Einstein tensors, respectively, as

$$\delta R_{\mu\nu}(h) = \frac{1}{2}[\Delta_L^{(2)}h_{\mu\nu} + \bar{\nabla}_\mu \bar{\nabla}_\nu h], \quad (20)$$

$$\delta R(h) = -\Lambda h, \quad (21)$$

$$\delta G_{\mu\nu}(h) = \frac{1}{2}\left[-\bar{\nabla}^2 h_{\mu\nu} + \bar{\nabla}_\mu \bar{\nabla}_\nu h + \frac{2\Lambda}{3}\left(h_{\mu\nu} + \frac{h}{2}\bar{g}_{\mu\nu}\right)\right]. \quad (22)$$

In order to find physically propagating modes, we decompose the metric perturbation $h_{\mu\nu}$ with 10 DOF covariantly into

$$h_{\mu\nu} = h_{\mu\nu}^{TT} + \bar{\nabla}_{(\mu} V_{\nu)} + \bar{\nabla}_\mu \bar{\nabla}_\nu \phi + \psi \bar{g}_{\mu\nu}, \quad (23)$$

where $h_{\mu\nu}^{TT}$ is the transverse traceless (TT) tensor with 5 DOF ($\bar{\nabla}^\mu h_{\mu\nu}^{TT} = 0, h^{TT} = 0$), V_ν is a divergence free vector with 3 DOF ($\bar{\nabla}^\mu V_\mu = 0$), and ϕ and ψ are scalar fields with 2 DOF. These imply two relations

$$\bar{\nabla}^2 h = \bar{\nabla}^4 \phi + 4\bar{\nabla}^2 \psi, \quad \bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} = \bar{\nabla}^4 \phi + \Lambda \bar{\nabla}^2 \phi + \bar{\nabla}^2 \psi. \quad (24)$$

One-particle scattering amplitude is mostly computed by choosing a condition of (19) even if one does not impose the gauge condition (18) [25]. Then, considering (24) together with this condition leads to

$$3\bar{\nabla}^2 \psi = \Lambda \bar{\nabla}^2 \phi \quad (25)$$

which implies that two scalars ϕ and ψ are not independent under the condition (19). Plugging this into the first relation of (24), one finds a relation between the trace h of $h_{\mu\nu}$ and scalar ψ as

$$h = \frac{3}{\Lambda}\left[\bar{\nabla}^2 + \frac{4\Lambda}{3}\right]\psi, \quad (26)$$

which implies that $\bar{\nabla}^2 + \frac{4\Lambda}{3}$ is not a physical pole and thus, it belongs to a removable (unphysical) pole [9]. Using (14), (21), and (26), we express ψ in terms of the trace T of external sources $T_{\mu\nu}$ as

$$\psi = \frac{1}{3\left[-2(\alpha + 3\beta)\bar{\nabla}^2 + 1\right]\left(\bar{\nabla}^2 + \frac{4}{3}\Lambda\right)}T \quad (27)$$

which means that ψ becomes a massive propagating scalar on AdS_4 space unless $\alpha = -3\beta$. However, imposing this condition of $\alpha = -3\beta$ to obtain critical gravity leads to that ψ becomes trivial.

To find the transverse traceless part $h_{\mu\nu}^{TT}$, we need to use the Lichnerowicz operator $\Delta_L^{(2)}$ acting on spin-2 symmetric tensors defined in (13). Taking into account the TT condition, we rewrite the linearized Einstein tensor as

$$\delta G_{\mu\nu}^{TT}(h) = \frac{1}{2}\Delta_L^{(2)}h_{\mu\nu}^{TT} - \Lambda h_{\mu\nu}^{TT}. \quad (28)$$

Then, we express $h_{\mu\nu}^{TT}$ in terms of external sources as

$$h_{\mu\nu}^{TT} = \frac{2}{\left[1 + \left(\frac{4\alpha}{3} + 8\beta\right)\Lambda + \alpha\bar{\nabla}^2\right](\Delta_L^{(2)} - 2\Lambda)} T_{\mu\nu}^{TT}, \quad (29)$$

where the transverse traceless source ($\bar{\nabla}^\mu T_{\mu\nu}^{TT} = 0, T^{TT} = 0$ with (15)) is given by [3]

$$T_{\mu\nu}^{TT} = T_{\mu\nu} - \frac{1}{3}T\bar{g}_{\mu\nu} + \frac{1}{3}\left[\frac{\bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{\Lambda}{3}\bar{g}_{\mu\nu}}{\bar{\nabla}^2 + \frac{4\Lambda}{3}}\right]T. \quad (30)$$

For the GR with CC, $h_{\mu\nu}^{TT}$ takes a simple form

$$h_{\mu\nu}^{TT} = \frac{2}{\Delta_L^{(2)} - 2\Lambda} T_{\mu\nu}^{TT}, \quad (31)$$

which for $\Lambda = 0$ gives us the gravitational wave [28]

$$h_{\mu\nu}^{TT} = -\frac{2}{\bar{\nabla}^2} T_{\mu\nu}^{TT} \quad (32)$$

in the Minkowski space.

We are now in a position to define the tree-level (one particle) exchange amplitude between two external sources $\tilde{T}_{\mu\nu}$ and $T_{\mu\nu}$ as

$$A = \frac{1}{4} \int d^4x \sqrt{-\bar{g}} \tilde{T}_{\mu\nu}(x) h^{\mu\nu}(x) \equiv \frac{1}{4} [\tilde{T}_{\mu\nu} h^{TT\mu\nu} + \tilde{T}\psi], \quad (33)$$

where we suppress the integral to have a notational simplicity in the last expression. Finally, the scattering amplitude takes the form [25]

$$\begin{aligned} 4A &= 2\tilde{T}_{\mu\nu} \left[\left(1 + \left(\frac{4\alpha}{3} + 8\beta\right)\Lambda + \alpha\bar{\nabla}^2\right)(\Delta_L^{(2)} - 2\Lambda) \right]^{-1} T^{\mu\nu} \\ &+ \frac{2}{3}\tilde{T} \left[\left(1 + \left(\frac{4\alpha}{3} + 8\beta\right)\Lambda + \alpha\bar{\nabla}^2\right)(\bar{\nabla}^2 + 2\Lambda) \right]^{-1} T \\ &- \frac{2\Lambda}{9}\tilde{T} \left[\left(1 + \left(\frac{4\alpha}{3} + 8\beta\right)\Lambda + \alpha\bar{\nabla}^2\right)(\bar{\nabla}^2 + 2\Lambda) \right]^{-1} \left[\bar{\nabla}^2 + \frac{4\Lambda}{3} \right]^{-1} T \\ &+ \frac{1}{3}\tilde{T} \left[-2(\alpha + 3\beta)\bar{\nabla}^2 + 1 \right]^{-1} \left[\bar{\nabla}^2 + \frac{4\Lambda}{3} \right]^{-1} T. \end{aligned} \quad (34)$$

We note that there is no vector V_μ in the scattering amplitude because it could be eliminated by choosing a further gauge-fixing [9].

3 vDVZ discontinuity in Minkowski space

In the case of $\Lambda = 0$, $m_0^2 \neq 0$, and $m_2^2 \neq 0$, the one-particle scattering amplitude (34) takes a simple form in the momentum space ($\nabla^2 \rightarrow -p^2$) as

$$\begin{aligned} [4A(p)]_{\text{Min}} &= \left[\tilde{T}_{\mu\nu} \frac{2}{p^2} T^{\mu\nu} - \tilde{T} \frac{1}{p^2} T \right] \\ &- \left[\tilde{T}_{\mu\nu} \frac{2}{p^2 + m_2^2} T^{\mu\nu} - \frac{2}{3} \tilde{T} \frac{1}{p^2 + m_2^2} T \right] \\ &+ \frac{1}{3} \tilde{T} \frac{1}{p^2 + m_0^2} T \end{aligned} \quad (35)$$

where two masses are given by

$$m_0^2 = \frac{1}{\alpha + 3\beta}, \quad m_2^2 = -\frac{1}{\alpha}. \quad (36)$$

In the convention of $\alpha < 0$ and $\beta > 0$, there is a massive ghost with positive mass squared ($m_2^2 > 0$) because of negative sign in the second line. Also, from the third line, one finds that a massive scalar is free from ghost and tachyon for $\alpha > -3\beta$. The amplitude (35) describes 8 DOF of 2 for massless graviton, 5 for massive graviton (ghost), and 1 for massive scalar [27]. In obtaining the second term -1 in the first line, we combine $-2/3$ obtained from $2/3p^2(p^2 + m_2^2)$ with $-1/3$ from $1/3p^2(p^2 + m_0^2)$. Residues to each pole in (35) are $\{[2, -1], [-2, \frac{2}{3}], \frac{1}{3}\}$ from top to bottom.

On the other hand, the corresponding Newtonian potential energy between two static sources $\tilde{T}_{\mu\nu} = M_1 \delta_\mu^0 \delta_\nu^0 \delta^3(\vec{x} - \vec{x}_1)$ and $T^{\mu\nu} = M_2 \delta_0^\mu \delta_0^\nu \delta^3(\vec{x} - \vec{x}_2)$ is given by

$$U(r) = \frac{GM_1 M_2}{r} \left[-\frac{1}{r} + \frac{4}{3} \frac{e^{-m_2 r}}{r} - \frac{1}{3} \frac{e^{-m_0 r}}{r} \right], \quad r = |\vec{x}_1 - \vec{x}_2|, \quad (37)$$

where the second term from ghost represents a repulsive force. The factors are clearly understood when considering $\tilde{T}_{\mu\nu} T^{\mu\nu} = \tilde{T}_{00} T_{00}$ and $\tilde{T} T = \tilde{T}_{00} T_{00}$. This shows why we started with external source $T_{\mu\nu}$.

Now we would like to mention the vDVZ discontinuity by pointing out a factor of $\frac{2}{3}$ at the last term in the second line of (35). In this case, the external source is necessary to compute residue and DOF. In order to avoid the ghost issue, one includes the Fierz-Pauli term instead of higher curvature terms in the bilinear Lagrangian as

$$I_{\text{FP}} = -\frac{M_{\text{FP}}^2}{4\kappa^2} \int d^4x \sqrt{-g} (h_{\mu\nu}^2 - h^2), \quad (38)$$

its amplitude leads to when choosing the gauge condition

$$[4A(p)]_{\text{FP}} = 2 \left[\tilde{T}_{\mu\nu} \frac{1}{p^2 + M_{\text{FP}}^2} T^{\mu\nu} - \frac{1}{3} \tilde{T} \frac{1}{p^2 + M_{\text{FP}}^2} T \right] \quad (39)$$

neglecting all contact terms (see (2.92) in Ref [29]). We call the former (latter) as the tensor (scalar) amplitudes, respectively. Residues to each pole in (39) are $[2, -\frac{2}{3}]$, which shows that the Fierz-Pauli massive gravity is free from the ghost. The residue “ $-\frac{2}{3}$ ” in (39) differs from “ -1 ” in the first line of (35), producing the famous vDVZ discontinuity in the massless limit of $M_{\text{FP}}^2 \rightarrow 0$. In general, this reflects a decreasing DOF of $5 \rightarrow 3$ in the limit of $M_{\text{FP}}^2 \rightarrow 0$. The same happens when replacing M_{FP}^2 by m_2^2 in the in the second line of (35) even it shows ghost states.

In order to investigate what happens in the massless limit of $M_{\text{FP}}^2 \rightarrow 0$ explicitly, we introduce the ADM formalism where the metric is parameterized as

$$ds_{\text{ADM}}^2 = -N^2 dt^2 + g_{ij} (dx^i - N^i dt) (dx^j - N^j dt). \quad (40)$$

Then, we have to consider the noncovariant perturbation around the Minkowski space as [30, 31]

$$g_{ij} = \delta_{ij} + h_{ij}, \quad N = 1 + n, \quad N_i = n_i \quad (41)$$

with

$$n = -\frac{1}{2}A, \quad n_i = \partial_i B + V_i, \quad h_{ij} = \psi \delta_{ij} + \partial_i \partial_j E + 2\partial_{(i} F_{j)} + t_{ij}. \quad (42)$$

Here the conditions of $\partial^i F_i = \partial^i V_i = \partial^i t_{ij} = t_{ii} = 0$ are imposed. The last two conditions mean that t_{ij} is a transverse and traceless tensor in three spatial dimensions. Using this decomposition, four scalar modes (A, B, ψ, E), two vector modes (V_i, F_i), and a tensor mode (t_{ij}) decouple completely from each other. These all amount to 10 DOF for a symmetric tensor in four dimensions. Considering the Fierz-Pauli mass term (38), we show that out of the 5 DOF of a massive graviton, 2 of these are expressed as transverse and traceless tensor modes t_{ij} , 2 of these are expressed as transverse vector modes $w_i = V_i - \dot{F}_i$, and the remaining one is from a scalar ψ . We introduce the external source term

$$S_{\text{int}} = -\frac{1}{2\kappa^2} \int dt d^3x \left[h^{ij} T_{ij} + 2h^{0j} T_{0j} + h^{00} T_{00} \right] \quad (43)$$

which leads to

$$S_{\text{int}} = -\frac{1}{2\kappa^2} \int dt d^3x \left[t_{ij} T_{ij} - 2w_i T_{0i} + \Phi T_{00} + \psi T_{ii} \right] \quad (44)$$

expressed in terms of gauge-invariant modes of ψ , $w_i = V_i - \dot{F}_i$, and $\Phi = A - 2\dot{B} + \ddot{E}$. Here Φ plays the role of Newtonian potential. The tensor equation takes a relatively simple form

$$t_{ij} = \frac{1}{\nabla^2 - M_{\text{FP}}^2} T_{ij}. \quad (45)$$

We find that there is no the vDVZ discontinuity in the massless limit because a single mass term M_{FP}^2 is present. This means that t_{ij} describes 2DOF in the massless limit as

$$t_{ij} = \frac{1}{\nabla^2} T_{ij}. \quad (46)$$

In the massless limit, a gauge-invariant vector reduces to

$$w_i = \frac{1}{\partial^2} T_{0i}, \quad \partial^2 = \partial_i \partial^i \quad (47)$$

which shows that the vector is a non-propagating mode because temporal derivative term ∂_0^2 is absent here. Finally, in the massless limit, ψ leads to

$$\psi = \frac{T_{ii} + 2T_{00} - \frac{3\tilde{T}_{00}}{\partial^2}}{6\nabla^2}, \quad (48)$$

which takes a further form

$$\psi = \frac{T_{00}}{2\partial^2} + \frac{T_{ii} - T_{00}}{6\nabla^2}. \quad (49)$$

This confirms the presence of the vDVZ discontinuity of the Fierz-Pauli mass term because the last term in (49) implies that ψ is a propagating degree of freedom.

Consequently, in the massless limit of $M_{\text{FP}}^2 \rightarrow 0$, total DOF is $3(=2+1)$ from t_{ij} and ψ , whereas ψ is absent in the GR. This shows the vDVZ discontinuity unambiguously in the Minkowski space. This is important because the extra scalar ψ predicts a value for the gravitational bending of light by a massive source that is $3/4$ of the Einstein prediction [3].

4 Residue calculation for $\alpha = -3\beta$

Choosing $\alpha = -3\beta$ for the non-critical gravity, the expression of (34) reduces to

$$\begin{aligned} [4A]_{\alpha=-3\beta} = & - 2m_2^2 \tilde{T}_{\mu\nu} \frac{1}{\left(\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2\right)(\Delta_L^{(2)} - 2\Lambda)} T^{\mu\nu} \\ & - \frac{2m_2^2}{3} \tilde{T} \frac{1}{\left(\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2\right)(\bar{\nabla}^2 + 2\Lambda)} T \\ & + \frac{2m_2^2 \Lambda}{9} \tilde{T} \frac{1}{\left(\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2\right)(\bar{\nabla}^2 + 2\Lambda)(\bar{\nabla}^2 + \frac{4\Lambda}{3})} T \\ & + \frac{1}{3} \tilde{T} \frac{1}{\bar{\nabla}^2 + \frac{4\Lambda}{3}} T. \end{aligned} \quad (50)$$

For $\Lambda \neq 0$, this is quite a non-trivial integral. Hence, we can study the particle spectrum of graviton and scalar by investigating each pole structure of their amplitude. Here we

read off three scalar poles ($\tilde{T}T$) from (50). Given these poles, finding their residues is easy. We wish to compute the residue at each pole.

(a) Pole at $\bar{\nabla}^2 = -\frac{4\Lambda}{3}$

The last two terms of (50) contribute to the scalar residue at this pole as

$$- \left[\frac{8\Lambda}{3m_2^2 + 8\Lambda} \right] \tilde{T}T. \quad (51)$$

The case of $\Lambda = 0$ leads to the zero residue. As was mentioned before, this corresponds to an unphysical pole.

(b) Pole at $\bar{\nabla}^2 = -2\Lambda$

The scalar residue at this physical pole takes the form

$$- \left[\frac{3m_2^2}{3m_2^2 + 10\Lambda} \right] \tilde{T}T. \quad (52)$$

As was pointed out previously, this pole may contain the gravitational waves in the Minkowski space limit for the Fierz-Pauli massive gravity. In order to recover it, we may take the limit of $\Lambda \rightarrow 0$. In this limit, the scalar residue is -1 , and thus, the amplitude describes a massless graviton with 2 DOF like

$$\lim_{\Lambda \rightarrow 0} [4A]_{\alpha=-3\beta} = 2 \left[\tilde{T}_{\mu\nu} \frac{1}{p^2} T^{\mu\nu} - \frac{1}{2} \tilde{T} \frac{1}{p^2} T \right], \quad (53)$$

which is the first line of (35).

(c) Pole at $\bar{\nabla}^2 = \frac{4\Lambda}{3} + m_2^2$

This is a new massive pole due to $\alpha R^{\mu\nu} R_{\mu\nu}$ -term. Its scalar residue takes the form

$$\left[\frac{2m_2^2(3m_2^2 + 7\Lambda)}{(3m_2^2 + 10\Lambda)(3m_2^2 + 8\Lambda)} \right]. \quad (54)$$

In the limit of $\Lambda \rightarrow 0$, it leads to $2/3$. However, one could not calculate the tensor residue unless one knows $\Delta_L^{(2)}$ explicitly. In order to compute it, one requires a further condition in the next section. This is a traceless condition of $h = 0$.

5 Residue with $\alpha = -3\beta$ and TT gauge

Requiring the traceless condition, we are working with the TT gauge. In this case, one finds that

$$\Delta_L^{(2)} h_{\mu\nu} = -\bar{\nabla}^2 h_{\mu\nu} + \frac{8\Lambda}{3} h_{\mu\nu}. \quad (55)$$

The scattering amplitude takes the form

$$\begin{aligned}
[4A]_{\alpha=-3\beta}^{TT} = \tilde{T}_{\mu\nu} h_{\mu\nu}^{TT} = & \quad 2m_2^2 \tilde{T}_{\mu\nu} \frac{1}{\left(\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2\right)\left(\bar{\nabla}^2 - \frac{2\Lambda}{3}\right)} T^{\mu\nu} \\
& - \frac{2m_2^2}{3} \tilde{T} \frac{1}{\left(\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2\right)(\bar{\nabla}^2 + 2\Lambda)} T \\
& + \frac{2m_2^2 \Lambda}{9} \tilde{T} \frac{1}{\left(\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2\right)(\bar{\nabla}^2 + 2\Lambda)(\bar{\nabla}^2 + \frac{4\Lambda}{3})} T.
\end{aligned} \tag{56}$$

Here are two tensor poles ($\tilde{T}_{\mu\nu} T^{\mu\nu}$) and four different scalar poles ($\tilde{T}T$).

(a) Pole at $\bar{\nabla}^2 = \frac{4\Lambda}{3} + m_2^2$

Comparing this pole with the AdS₄ pole ($\bar{\nabla}^2 - \frac{2\Lambda}{3}$), it requires an inequality for m_2^2 as [11]

$$0 < m_2^2 < -\frac{2\Lambda}{3}, \tag{57}$$

where the saturation of right bound implies the critical gravity. This bound for non-critical gravity is necessary to discuss the vDVZ discontinuity in AdS₄ space. Similarly, one has a bound of $0 < M_{\text{FP}}^2 < 2\Lambda/3$ in the dS₄ space [1]. We note that the critical gravity of $m_2^2 = -2\Lambda/3$ is beyond the above bound. Its residue is calculated to be

$$- \left[\frac{2m_2^2}{m_2^2 + \frac{2\Lambda}{3}} \right] \tilde{T}_{\mu\nu} T^{\mu\nu} + \left[\frac{2m_2^2(3m_2^2 + 7\Lambda)}{(3m_2^2 + 10\Lambda)(3m_2^2 + 8\Lambda)} \right] \tilde{T}T. \tag{58}$$

Taking first the $m_2^2 \rightarrow 0$ limit and then, the $\Lambda \rightarrow 0$ limit leads to $[0, 0]$, which means the zero residues. However, one has the scalar residue

$$\left[\frac{2\Lambda - 2M_{\text{FP}}^2}{3M_{\text{FP}}^2 - 2\Lambda} \right] \tilde{T}T \tag{59}$$

for the Fierz-Pauli massive gravity in AdS₄ space [3]. In the limit of the $M_{\text{FP}}^2 \rightarrow 0$ first, it leads to -1 (no vDVZ discontinuity), while for the $\Lambda \rightarrow 0$ limit first, it takes a form of residue $-2/3$ (vDVZ discontinuity). Hence, this case of zero residue is special when comparing with the Fierz-Pauli massive gravity in AdS₄ space.

On the other hand, taking first the $\Lambda \rightarrow 0$ limit and then, the $m_2^2 \rightarrow 0$ limit leads to the residues of $[-2, \frac{2}{3}]$, which means that the vDVZ discontinuity occurs in the ghost expression. Especially, at the critical point of $m_2^2 = -2\Lambda/3$, the first term blows up ($-\infty$), while the second term gives us its residue of $-5/36$. This means that it is impossible for the critical gravity to have a finite tensor residue. This is a new feature of residue of $\frac{1}{p^2 - \frac{4\Lambda}{3} + m_2^2}$ pole at the critical point.

(b) Pole at $\bar{\nabla}^2 = \frac{2\Lambda}{3}$

The tensor residue is calculated to be

$$\left[\frac{2m_2^2}{m_2^2 + \frac{2\Lambda}{3}} \right] \tilde{T}_{\mu\nu} T^{\mu\nu}, \quad (60)$$

which blows up (∞) for the critical case of $m_2^2 = -2\Lambda/3$. This can be confirmed by rewriting the first line of (56) as

$$[4A]_{(\alpha=-3\beta)}^{TT} \rightarrow \left[\frac{2m_2^2}{m_2^2 + \frac{2\Lambda}{3}} \right] \tilde{T}_{\mu\nu} \left[\frac{1}{\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2} - \frac{1}{\bar{\nabla}^2 - \frac{2\Lambda}{3}} \right] T^{\mu\nu}. \quad (61)$$

For the residue calculation, its momentum space representation takes the form

$$[4A(p)]_{(\alpha=-3\beta)}^{TT} \rightarrow \left[\frac{2m_2^2}{m_2^2 + \frac{2\Lambda}{3}} \right] \tilde{T}_{\mu\nu} \left[-\frac{1}{p^2 + \frac{4\Lambda}{3} + m_2^2} + \frac{1}{p^2 + \frac{2\Lambda}{3}} \right] T^{\mu\nu}. \quad (62)$$

This shows a massive graviton with negative energy (ghost).

c) Pole at $\bar{\nabla}^2 = -\frac{4\Lambda}{3}$

The scalar residue at this pole is

$$\left[\frac{m_2^2}{3m_2^2 + 8\Lambda} \right] \tilde{T}T \quad (63)$$

which is contributed from the last term of (56) only. The case of $\Lambda = 0$ shows its residue of $1/3$. This corresponds to an unphysical pole.

(d) Pole at $\bar{\nabla}^2 = -2\Lambda$

The scalar residue at this pole takes the form

$$- \left[\frac{3m_2^2}{3m_2^2 + 10\Lambda} \right] \tilde{T}T. \quad (64)$$

This pole may contain the gravitational waves in the Minkowski space limit. In order to recover it, we may take the limit of $\Lambda \rightarrow 0$. In this limit, the residue is -1 , and thus, the amplitude describes a massless graviton with 2 DOF like as in the first line of (35).

On the other hand, without external source, equation (9) satisfies the fourth-order equation under the TT gauge [11]

$$\left(\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2 \right) \left(\bar{\nabla}^2 - \frac{2\Lambda}{3} \right) h_{\mu\nu} = 0 \quad (65)$$

which is obtained by taking into account the first line of (56) only. Hence, one did not consider the scalar amplitudes of the last two terms. For $m_2^2 \neq -2\Lambda/3$, it implies a massless graviton satisfying the second-order equation

$$\left(\bar{\nabla}^2 - \frac{2\Lambda}{3} \right) h_{\mu\nu}^m = 0 \quad (66)$$

and a massive graviton satisfying the second-order equation

$$\left(\bar{\nabla}^2 - \frac{4\Lambda}{3} - m_2^2\right)h_{\mu\nu}^M = 0. \quad (67)$$

For $m_2^2 = -2\Lambda/3$, (67) degenerates (66), which defines the critical gravity. Precisely speaking, the fourth-order equation (65) cannot be split into two second-order equations (66) and (67) for $m_2^2 = -2\Lambda/3$. Its promising form is the fourth-order equation

$$\left(\bar{\nabla}^2 - \frac{2\Lambda}{3}\right)^2 h_{\mu\nu}^{\text{cr}} = 0 \quad (68)$$

for the critical gravity. The solution is given by the log-gravity [14]

$$h_{\mu\nu}^{\text{cr}} = h_{\mu\nu}^{\text{log}} = f(t, \rho) h_{\mu\nu}^m \quad (69)$$

where

$$f(t, \rho) = 2it + \ln[\sinh(2\rho)] - \ln[\tanh \rho]. \quad (70)$$

However, the non-unitarity issue of log-gravity is not still resolved, indicating that the log gravity suffers from the ghost problem [15, 16].

6 Discussions

We start by noting that in the FP massive gravity, its scalar residue of $\frac{1}{p^2 + M_{\text{FP}}^2}$ pole is $-2/3$ in the Minkowski space, showing the vDVZ discontinuity instead of -1 for the massless case. In the AdS_4 space, its scalar residue of $\frac{1}{p^2 - 2\Lambda + M_{\text{FP}}^2}$ pole in (59) is -1 for taking the $M_{\text{FP}}^2 \rightarrow 0$ limit first, while its scalar residue is given by $-2/3$ for taking the $\Lambda \rightarrow 0$ limit first.

In the higher curvature gravity, the scalar residue of $\frac{1}{p^2 + m_2^2}$ pole is $2/3$ in the Minkowski space because of the ghost state. On the other hand, for non-critical gravity with $0 < m_2^2 < -2\Lambda/3$, the scalar residue of $\frac{1}{p^2 + \frac{4\Lambda}{3} + m_2^2}$ pole is 0 for taking the $m_2^2 \rightarrow 0$ limit first, while its scalar residue is $2/3$ for taking the $\Lambda \rightarrow 0$ limit first because of the ghost state. The latter represents that propagating DOF is decreased from 5 to 3 in the massless limit of $m_2^2 \rightarrow 0$.

Considering that the vDVZ discontinuity is the massless limit of the massive gravity in the Minkowski space, the critical gravity appears as the massless limit of higher curvature gravity in the AdS_4 space. Hence, we proposed that the critical gravity may be represented as the vDVZ discontinuity of the higher curvature gravity in the AdS_4 space. In this case, we expect to have a decreasing DOF ($5 \rightarrow 3$) at the critical point. However, at the critical

point of $m_2^2 = -2\Lambda/3$, the tensor residue of $\frac{1}{p^2 + \frac{4\Lambda}{3} + m_2^2}$ blows up, while its scalar residue is $-5/36$. This is surely an unpromising feature of the critical gravity when plugging external sources.

Finally, we wish to comment on how to understand the log-gravity solution at the critical point when plugging the external source. In our approach, the conditions of transverse and traceless were taken into account by the external source, while these conditions were used to derive (68) and, obtain the massless modes $h_{\mu\nu}^m$ and its log-modes $h_{\mu\nu}^{\log}$. Even though the log-modes has positive excitation energy, it was pointed out that negative norm states (ghost) appears from linear combination of massless and log-modes [15]. Eventually, the ghost-free condition requires the truncation of log-modes by choosing an appropriate boundary condition [11, 17]. However, this means that all massive modes are eliminated from the IR region, leaving with the Einstein gravity. In this case, the role of higher curvature terms is unclear at the linearized level.

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